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Empirical Algebraic Geometry

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In a previous paper we introduced the notion of an orthogonal category and generalized the notion of a sheaf of sets on a complete Boolean algebra **B** to that of a sheaf on the complete Boolean algebra **B** with values in an orthogonal category $\mathfrak{K}$. By properly replacing the complete Boolean algebra **B** by a manual $\mathfrak{M}$ of Boolean locales, we get a notion of a sheaf on $\mathfrak{M}$ with values in $\mathfrak{K}$, which can be regarded as a quantum generalization of a sheaf on **B**. Taking $\mathfrak{K}$ to be the category of sheaves of Abelian groups or that of schemes à la Grothendieck, we will discuss some fundamental aspects of the quantum generalizations of sheaves and schemes.

INTRODUCTION

As we have shown (Nishimura, 1993), the garments of category theory liberalize the notion of a manual introduced by Foulis and Randall (1972; Randall and Foulis, 1973). The resulting mathematical structure is called a manual of Boolean locales, which is a small subcategory of the dual category $\mathfrak{BLoc}$ of the category $\mathfrak{Bool}$ of complete Boolean algebras and complete Boolean homomorphisms with mild constraints. We hold that the logical aspect of the operational foundations of empirical sciences, including quantum mechanics in particular, is suitably represented by a manual of Boolean locales.

In a previous paper (Nishimura, 1995) we discussed the notion of a sheaf on a complete Boolean algebra **B** with values in an orthogonal category $\mathfrak{K}$. Since Boolean mathematics has been concerned with sheaves on a complete Boolean algebra and we believe that the logic is to be represented not necessarily by a complete Boolean algebra but generally by a manual of Boolean locales, it is not unnatural that we should be led to the notion of a sheaf on a manual $\mathfrak{M}$ of Boolean locales with values in an orthogonal category $\mathfrak{K}$,

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which will be presented in Section 1. Since the category of sheaves of Abelian groups and that of schemes are naturally rendered orthogonal categories with coproduct diagrams as orthogonal sum diagrams, we can discuss a sheaf of sheaves of Abelian groups on $\mathfrak{M}$ and a sheaf of schemes on $\mathfrak{M}$ in Sections 3 and 4, respectively. Section 2 is devoted to preliminary considerations on sheaves of Abelian groups.

In this paper a ring always means a cummutative ring with identity. A homomorphism of rings is always required to preserve identities. A prime ideal I of a ring R is always required to be $I \neq R$.

1. O-SHEAVES

Let us begin this section with a brief review of the notion of an orthogonal category introduced by Nishimura (1995). A pair $(\mathfrak{K}, \mathfrak{oS}_{\mathfrak{K}})$ of a category $\mathfrak{K}$ and a class $\mathfrak{oS}_{\mathfrak{K}}$ of diagrams in $\mathfrak{K}$ is called an *orthogonal category* if it satisfies the following conditions:

(1.1) The category $\mathfrak{K}$ has an initial object.

(1.2) Every diagram in $\mathfrak{oS}_{\mathfrak{K}}$ is of the form $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$.

(1.3) For any small family $\{\mathbf{X}\}_{\lambda\in\Lambda}$ of objects in $\mathfrak{K}$ there exist an object $\mathbf{Y}$ in $\mathfrak{K}$ and a family $\{\mathbf{f}_\lambda\}_{\lambda\in\Lambda}$ of morphisms $\mathbf{f}_\lambda$: $\mathbf{X}_\lambda \to \mathbf{Y}$ in $\mathfrak{K}$ such that the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$.

(1.4) Given a small family $\{\mathbf{X}_\lambda\}_{\lambda\in\Lambda}$ of objects in $\mathfrak{K}$, if diagrams $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ and $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{g}_\lambda} \mathbf{Z}\}_{\lambda\in\Lambda}$ lie in $\mathfrak{oS}_{\mathfrak{K}}$, then there exists a unique morphism $\mathbf{h}$: $\mathbf{Y} \to \mathbf{Z}$ in $\mathfrak{K}$ such that $\mathbf{g}_\lambda = \mathbf{h} \circ \mathbf{f}_\lambda$ for each $\lambda \in \Lambda$.

(1.5) Given diagrams $\{\mathbf{Y}_\lambda \xrightarrow{\mathbf{g}_\lambda} \mathbf{Z}\}_{\lambda\in\Lambda}$ and $\{\mathbf{X}_\delta \xrightarrow{\mathbf{f}_\delta} \mathbf{Y}_\lambda\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ in $\mathfrak{K}$, the diagram $\{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\lambda \circ \mathbf{f}_\delta} \mathbf{Z} | _{\lambda\in\Lambda}$ and $\delta \in \Delta_\lambda\}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$ iff all the diagrams $\{\mathbf{Y}_\lambda \xrightarrow{\mathbf{g}_\lambda} \mathbf{Z}\}_{\lambda\in\Lambda}$ and $\{\mathbf{X}_\delta \xrightarrow{\mathbf{f}_\delta} \mathbf{Y}_\lambda\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ lie in $\mathfrak{oS}_{\mathfrak{K}}$, where the sets Δ_λ are assumed to be mutually disjoint.

(1.6) If a diagram $\{\mathbf{X}_\delta \xrightarrow{\mathbf{f}_\delta} \mathbf{Y} | \lambda \in \Lambda$ and $\delta \in \Delta_\lambda\}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$, then there exist diagrams $\{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Z}_\lambda\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ and $\{\mathbf{Z}_\lambda \xrightarrow{\mathbf{h}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ such that $\mathbf{f}_\delta = \mathbf{h}_\lambda \circ \mathbf{g}_\delta$ for any $\lambda \in \Lambda$ and any $\delta \in \Delta_\lambda$, where the sets Δ_λ are assumed to be mutually disjoint.

(1.7) If $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ is a diagram in $\mathfrak{K}$ and $\{\mathbf{Z}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta\in\Delta}$ is also a diagram in $\mathfrak{K}$ with $\mathbf{Z}_\delta$ being an initial object of $\mathfrak{K}$ for each $\delta \in \Delta$, then the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ is in $\mathfrak{oS}_{\mathfrak{K}}$ iff the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda} \cup \{\mathbf{Z}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta\in\Delta}$ is in $\mathfrak{oS}_{\mathfrak{K}}$.

(1.8) If $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is an isomorphism in $\mathfrak{K}$, then the diagram $\{\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y}\}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$.

(1.9) Given a diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ in $\mathfrak{oS}_{\mathfrak{K}}$, if $\mathbf{f}_{\lambda_1}$ and $\mathbf{f}_{\lambda_2}$ happen to be the same morphism for some distinct $\lambda_1, \lambda_2 \in \Lambda$ (so that $\mathbf{X}_{\lambda_1} = \mathbf{X}_{\lambda_2}$), then $\mathbf{X}_{\lambda_1} = \mathbf{X}_{\lambda_2}$ is an initial object of $\mathfrak{K}$.

(1.10) If a diagram $\{\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y}\}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$, then $\mathbf{f}$ is an isomorphism.

(1.11) Given diagrams $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ and $\{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta\in\Delta}$ in $\mathfrak{K}$, if both the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ and the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda} \cup \{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta\in\Delta}$ are in $\mathfrak{oS}_{\mathfrak{K}}$, then $\mathbf{X}_\delta$ is an initial object for each $\delta \in \Delta$.

Unless confusion may arise, the category $\mathfrak{K}$ itself is called an *orthogonal category* by abuse of language. A diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ in $\mathfrak{oS}_{\mathfrak{K}}$ is called an *orthogonal sum diagram*, in which $\mathbf{Y}$ is called an *orthogonal* sum of $\mathbf{X}_\lambda$'s and is denoted by $\Sigma_{\lambda\in\Lambda} \oplus \mathbf{X}_\lambda$. Thus the class $\mathfrak{oS}_{\mathfrak{K}}$ is the class of orthogonal sum diagrams in $\mathfrak{K}$. A morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is called an *embedding* if there exists a morphism $\mathbf{g}$: $\mathbf{Z} \to \mathbf{Y}$ in $\mathfrak{K}$ such that the diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{\mathbf{g}} \mathbf{Z}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$. Two embeddings $\mathbf{f}$: $\mathbf{Y} \to \mathbf{X}$ and $\mathbf{g}$: $\mathbf{Z} \to \mathbf{X}$ with the same codomain are said to be *equivalent* if there exists an isomorphism $\mathbf{h}$: $\mathbf{Y} \to \mathbf{Z}$ in $\mathfrak{K}$ such that $\mathbf{f} = \mathbf{g} \circ \mathbf{h}$. An object in $\mathfrak{K}$ is called *trivial* if it is an initial object of $\mathfrak{K}$. A trivial object of $\mathfrak{K}$ can be regarded as the orthogonal sum of the empty family of objects in $\mathfrak{K}$.

Example 1.1. The category $\mathfrak{Bool}$ of complete Boolean algebras and complete Boolean homomorphisms is a complete category, so that the dual category $\mathfrak{BLoc}$ of $\mathfrak{Bool}$ is cocomplete. It is easy to see that the pair ($\mathfrak{BLoc}$, $\mathfrak{cp}_{\mathfrak{BLoc}}$) is an orthogonal category, where $\mathfrak{cp}_{\mathfrak{BLoc}}$ denotes the class of coproduct diagrams in $\mathfrak{BLoc}$.

Example 1.2. It is well known that the category of $\mathfrak{Top}$ of topological spaces and continuous mappings is a complete and cocomplete category. It is easy to see that the pair ($\mathfrak{Top}$, $\mathfrak{cp}_{\mathfrak{Top}}$) is an orthogonal category, where $\mathfrak{cp}_{\mathfrak{Top}}$ denotes the class of coproduct diagrams in $\mathfrak{Top}$.

Example 1.3. It is well known that the category $\mathfrak{Rng}$ of rings and homomorphisms is a complete category, so that the dual category $\mathfrak{RLoc}$ of $\mathfrak{Rng}$ is a cocomplete category. It is easy to see that the pair ($\mathfrak{RLoc}$, $\mathfrak{cp}_{\mathfrak{RLoc}}$) is an orthogonal category.

A functor $\mathfrak{F}$ from an orthogonal cagegory $\mathfrak{K}$ to another orthogonal category $\mathfrak{L}$ is called an O-*functor* if it maps orthogonal diagrams in $\mathfrak{K}$ to orthogonal diagrams in $\mathfrak{L}$.

Let $\mathfrak{M}$ be a small subcategory of an orthogonal category $\mathfrak{K}$. A diagram in $\mathfrak{K}$ is said to be *in* $\mathfrak{M}$ if all the objects and morphisms occurring in the diagram lie in $\mathfrak{M}$. Objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$ are said to be $\mathfrak{M}$-*orthogonal*, in notation $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, if there exists an orthogonal sum diagram

$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$. An object of $\mathfrak{M}$ is said to be *$\mathfrak{M}$-trivial* if it is a trivial object of $\mathfrak{K}$ and also an initial object of $\mathfrak{M}$. An object $\mathbf{X}$ of $\mathfrak{M}$ is said to be *$\mathfrak{M}$-maximal* if for any object $\mathbf{Y}$ of $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ implies that $\mathbf{Y}$ is $\mathfrak{M}$-trivial. Objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$ are said to be *$\mathfrak{M}$-equivalent*, in notation $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$, provided that for any objects $\mathbf{Z}$ of $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$. Obviously $\mathfrak{M}$-equivalence is an equivalence relation among the objects of $\mathfrak{M}$. We denote by $[\mathbf{X}]_{\mathfrak{M}}$ the equivalence class of an object $\mathbf{X}$ of $\mathfrak{M}$ with respect to $\mathfrak{M}$-equivalence. An orthogonal sum diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$ is said to be an *orthogonal $\mathfrak{M}$-sum diagram* if for any orthogonal sum diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}'_\lambda} \mathbf{X}'\}_{\lambda \in \Lambda}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$ the unique morphism $\mathbf{g}$: $\mathbf{X} \to \mathbf{X}'$ of $\mathfrak{K}$ with $\mathbf{g} \circ \mathbf{f}_\lambda = \mathbf{f}'_\lambda$ for any $\lambda \in \Lambda$ belongs to $\mathfrak{M}$, in which $\mathbf{X}$ is called an *orthogonal* $\mathfrak{M}$-sum of $\mathbf{X}_\lambda$'s and is denoted by $\Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$. If Λ is a finite set, say $\Lambda = \{1, 2\}$, then such a notation as $\mathbf{X}_1 \oplus_{\mathfrak{M}} \mathbf{X}_2$ is preferred. Note that an $\mathfrak{M}$-trivial object of $\mathfrak{M}$, if it exists, can be regarded as an orthogonal $\mathfrak{M}$-sum of the empty family of objects of $\mathfrak{M}$. A morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is called an *$\mathfrak{M}$-embedding* if there exists a morphism $\mathbf{g}$: $\mathbf{Z} \to \mathbf{Y}$ such that the diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{\mathbf{g}} \mathbf{Z}$ is an orthogonal $\mathfrak{M}$-sum diagram. Given objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$, if there exists an $\mathfrak{M}$-embedding $\mathbf{f}$: $\mathbf{X} \to \mathbf{X}$ in $\mathfrak{M}$, then we say that $\mathbf{X}$ is an *$\mathfrak{M}$-subobject of* $\mathbf{Y}$.

Given an orthogonal category $\mathfrak{K}$, a *manual in* $\mathfrak{K}$ or an *$\mathfrak{K}$-manual* for short is a small subcategory of $\mathfrak{K}$ abiding by the following conditions:

(1.12) For any pair $(\mathbf{X}, \mathbf{Y})$ of objects in $\mathfrak{M}$, there exists at most a sole morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$.

(1.13) There exists at least a trivial object of $\mathfrak{K}$ in $\mathfrak{M}$.

(1.14) Every trivial object of $\mathfrak{K}$ in $\mathfrak{M}$ is $\mathfrak{M}$-trivial.

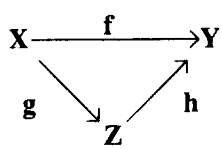
(1.15) For any objects $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, if there exists a morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ for any object $\mathbf{Z}$ in $\mathfrak{M}$.

(1.16) For any objects $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, there exists an object $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{K}$.

(1.17) For any object $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{W}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{W}$ imply $\mathbf{Z} \perp_{\mathfrak{M}} \mathbf{W}$ for any object $\mathbf{W}$ in $\mathfrak{M}$.

(1.18) For any objects $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$ iff there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, and both of $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ are $\mathfrak{M}$-maximal.

(1.19) For any commutative diagram



of $\mathfrak{K}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{h}$ is an $\mathfrak{M}$-embedding, then $\mathbf{g}$ is in $\mathfrak{M}$.

A $\mathfrak{K}$-manual $\mathfrak{M}$ is said to be *rich* if it satisfies the following condition:

(1.20) For any object $\mathbf{X}$ in $\mathfrak{M}$ and any embedding $\mathbf{f}$: $\mathbf{Y} \to \mathbf{X}$ in $\mathfrak{K}$, there exists an $\mathfrak{M}$-embedding $\mathbf{f}'$: $\mathbf{Y}' \to \mathbf{X}$ in $\mathfrak{M}$ such that $\mathbf{f}$ and $\mathbf{f}'$ are equivalent in $\mathfrak{K}$.

The following proposition is proved in Nishimura (1995).

Proposition 1.3. For any finite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, $\Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ exists.

A manual $\mathfrak{M}$ in an orthogonal category $\mathfrak{K}$ is called *σ-coherent* or *completely coherent* if it satisfies condition $(1.21)_\sigma$ or $(1.21)_\infty$, respectively:

$(1.21)_\sigma$ For any sequence $\{\mathbf{X}_i\}_{i \in \mathbf{N}}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{Z} = \Sigma_{i \in \mathbf{N}} \oplus_{\mathfrak{M}} \mathbf{X}_i$.

$(1.21)_\infty$ For any infinite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ with $\mathbf{Z} = \Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$.

In the remainder of this paper a *manual of Boolean locales* always means a completely coherent rich manual in the orthogonal category ($\mathfrak{BLoc}$, $\mathrm{cc}_{\mathfrak{BLoc}}$). A manual $\mathfrak{M}$ of Boolean locales is said to be *finite* if any object $\mathbf{X}$ in $\mathfrak{M}$, regarded as a Boolean algebra, consists of a finite number of elements.

Given a manual $\mathfrak{M}$ of Boolean locales and an orthogonal category $\mathfrak{K}$, a functor $\mathfrak{F}$: $\mathfrak{M} \to \mathfrak{K}$ is called an O-*sheaf* on $\mathfrak{M}$ with values in $\mathfrak{K}$ if it maps orthogonal $\mathfrak{M}$-sum diagrams to orthogonal diagrams in $\mathfrak{K}$. Given two O-sheaves of $\mathfrak{F}$ and $\mathfrak{G}$ on $\mathfrak{M}$ with values in $\mathfrak{K}$, a morphism from $\mathfrak{F}$ to $\mathfrak{G}$ is defined to be a natural transformation from the functor $\mathfrak{F}$ to the functor $\mathfrak{G}$. The totality of O-sheaves on $\mathfrak{M}$ with values in $\mathfrak{K}$ and morphisms between them forms a category, denoted by $\mathbb{O}\text{=}\mathbb{Sh}_{\mathfrak{M}}(\mathfrak{K})$ or by $\mathbb{O}\text{=}\mathbb{Sh}(\mathfrak{K})$ if $\mathfrak{M}$ is explicit from the context.

Example 1.5. Given a complete Boolean algebra $\mathbf{B}$ and an orthogonal category $\mathfrak{K}$, the notion of a $\mathfrak{K}$-sheaf on $\mathbf{B}$ discussed in Nishimura (1995), if $\mathbf{B}$ is identified with the first-class Boolean manual $\mathfrak{M}_{\mathbf{B}}$ on $\mathbf{B}$, is no other than our present notion of an O-sheaf on $\mathfrak{M}_{\mathbf{B}}$ with values in $\mathfrak{K}$.

The following proposition should be obvious.

Proposition 1.6. Given a manual $\mathfrak{M}$ of Boolean locales and an O-functor $\mathfrak{H}$: $\mathfrak{K} \to \mathfrak{L}$ between orthogonal categories, if $\mathfrak{F}$ is an O-sheaf on $\mathfrak{M}$ with values in $\mathfrak{K}$, then $\mathfrak{H} \circ \mathfrak{F}$ is an O-sheaf on $\mathfrak{M}$ with values in $\mathfrak{L}$. Thus $\mathfrak{H}$ naturally induces a functor from $\mathbb{O}\text{=}\mathbb{Sh}_{\mathfrak{M}}(\mathfrak{K})$ to $\mathbb{O}\text{=}\mathbb{Sh}_{\mathfrak{M}}(\mathfrak{L})$.

Corollary 1.7. Given a finite manual $\mathfrak{M}$ of Boolean locales and a functor $\mathfrak{H}$: $\mathfrak{K} \to \mathfrak{L}$ between orthogonal categories mapping finite orthogonal sum diagrams in $\mathfrak{K}$ to finite orthogonal sum diagrams in $\mathfrak{L}$, if $\mathfrak{F}$ is an O-sheaf on $\mathfrak{M}$ with values in $\mathfrak{K}$, then $\mathfrak{H} \circ \mathfrak{F}$ is an O-sheaf on $\mathfrak{M}$ with values in $\mathfrak{L}$. Thus $\mathfrak{H}$ naturally induces a functor from $\mathbb{O}{=}\mathbb{Sh}_{\mathfrak{M}}(\mathfrak{K})$ to $\mathbb{O}{=}\mathbb{Sh}_{\mathfrak{M}}(\mathfrak{L})$.

2. PRELIMINARY CONSIDERATIONS ON SHEAVES

Given a topological space **X**, we denote by $\mathbf{Sh}_{ab}(X)$ the category of sheaves of Abelian groups and homomorphisms between them. A continuous function $f: X \to Y$ of topological spaces has two associated functors f_*: $\mathbf{Sh}_{ab}(X) \to \mathbf{Sh}_{ab}(Y)$ and f^*: $\mathbf{Sh}_{ab}(Y) \to \mathbf{Sh}_{ab}(X)$. The functor f_* assigns to each sheaf $\mathscr{A}$ of Abelian groups on X of its direct image sheaf $f_*\mathscr{A}$ on Y, while the functor f^* assigns to each sheaf $\mathscr{B}$ of Abelian groups on Y of its inverse image sheaf $f^*\mathscr{B}$ on X. As is well known, we have the canonical adjunction

$$\mathrm{Hom}_X\,(f^*\mathscr{B}, \mathscr{A}) \cong \mathrm{Hom}_Y(\mathscr{B}, f_*\mathscr{A})$$

for a sheaf $\mathscr{A}$ of Abelian groups on X and a sheaf $\mathscr{B}$ of Abelian groups on Y, for which the reader is referred, e.g., to Hartshorne (1977, Chapter II, Exercise 1.18). The unit and the counit of the adjunction are denoted by η_f and ϵ_f, respectively.

Proposition 2.1. The assignment to each topological space X of the category $\mathbf{Sh}_{ab}(X)$ and to each continuous mapping f: $X \to Y$ of topological spaces of the standard adjunction

$$\mathbf{Sh}_{ab}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Sh}_{ab}(X)$$

is a contravariant functor from the category $\mathfrak{Top}$ to the category $\mathfrak{Adj}$, where $\mathfrak{Adj}$ denotes the category whose objects are categories and whose morphisms are adjunctions (MacLane, 1971, Chapter IV, §8).

Corollary 2.2. Given a commutative square of continuous functions

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha_1} & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_2 & \xrightarrow[\alpha_2]{} & Y_2 \end{array}$$

and sheaves of Abelian groups $\mathscr{A}_1$ and $\mathscr{B}_2$ on X_1 and Y_2, respectively, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{X_1}(\alpha_1^* g^* \mathcal{B}_2, \mathcal{A}_1) & \cong & \mathrm{Hom}_{Y_1}(g^* \mathcal{B}_2, \alpha_{1*}\mathcal{A}_1) \\
\| & & \wr\| \\
\mathrm{Hom}_{X_1}(f^* \alpha_2^* \mathcal{B}_2, \mathcal{A}_1) & & \mathrm{Hom}_{Y_2}(\mathcal{B}_2, g_* \alpha_{1*}\mathcal{A}_1) \\
\wr\| & & \| \\
\mathrm{Hom}_{X_2}(\alpha_2^* \mathcal{B}_2, f_* \mathcal{A}_1) & \cong & \mathrm{Hom}_{Y_2}(\mathcal{B}_2, \alpha_{2*} f^* \mathcal{A}_1)
\end{array}$$

3. EMPIRICAL SHEAF THEORY

We will define the category $\mathfrak{Sh}_{\mathfrak{ab}}$ of sheaves of Abelian groups. The objects of $\mathfrak{Sh}_{\mathfrak{ab}}$ are all pairs $(X, \mathcal{A})$ of a topological space X and a sheaf $\mathcal{A}$ of Abelian groups on X. Given sheaves of Abelian groups $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, the morphisms from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ in $\mathfrak{Sh}_{\mathfrak{ab}}$ are all pairs $(f, f^\#)$ of a continuous function f: $X \to Y$ and a homomorphism $f^\#$: $\mathcal{B} \to f_*\mathcal{A}$ of sheaves of Abelian groups on Y. By dint of adjunction $\mathrm{Hom}_X(f^*\mathcal{B}, \mathcal{A}) \cong \mathrm{Hom}_Y(\mathcal{B}, f_*\mathcal{A})$, the morphism $(f, f^\#)$: $(X, A) \to (Y, \mathcal{B})$ in $\mathfrak{Sh}_{\mathfrak{ab}}$ can be represented also by $(f, f_\#)$, where $f_\#$ is the homomorphism of sheaves of Abelian groups on X corresponding to $f^\#$ under the above adjunction. The corresponding $(f, f^\#)$ and $(f, f_\#)$ are respectively called the *upper* and *lower representations* of the same morphism of $\mathfrak{Sh}_{\mathfrak{ab}}$. We will use $(f, f^\#)$ and $(f, f_\#)$ interchangeably according to context. If the underlying continuous function f: $X \to Y$ is an identity function (so that $X = Y$), then the upper and lower representations coincide (i.e., $f^\# = f_\#$). Given morphisms $(f, f^\#)$: $(X, \mathcal{A}) \to (Y, \mathcal{B})$ and $(g, g^\#)$: $(Y, \mathcal{B}) \to (Z, \mathcal{C})$ in $\mathfrak{Sh}_{\mathfrak{ab}}$, their composition $(g, g^\#) \circ (f, f^\#)$ is defined to be $(g \circ f, (g_* f^\#) \circ g^\#)$. As for the lower representation of composition of morphisms in $\mathfrak{Sh}_{\mathfrak{ab}}$, we have the following result.

Proposition 3.1. If $(f, f^\#)$ and $(g, g^\#)$ are represented lowerly by $(f, f_\#)$ and $(g, g_\#)$, respectively, then their composition $(g, g_\#) \circ (f, f_\#)$ is representable lowerly as $(g \circ f, f_\# \circ (f^* g_\#))$.

Proof. By chasing $f_\#$ around the commutative square

$$\begin{array}{ccccc}
 & \mathrm{Hom}_X(f^* g^* \mathcal{C}, \mathcal{A}) & \cong & \mathrm{Hom}_Y(g^* \mathcal{C}, f_* \mathcal{A}) & \\
\mathrm{Hom}_X(f^* g_\#, \mathcal{A}) & \uparrow & & \uparrow & \mathrm{Hom}_Y(g_\#, f_* \mathcal{A}) \\
 & \mathrm{Hom}_X(f^* \mathcal{B}, \mathcal{A}) & \cong & \mathrm{Hom}_Y(\mathcal{B}, f_* \mathcal{A}) &
\end{array}$$

we have

$$\begin{array}{ccc} f_{\#}\circ(f^{*}g_{\#}) & \longmapsto & f^{\#}\circ g_{\#} \\ \updownarrow & & \updownarrow \\ f_{\#} & \longmapsto & f^{\#} \end{array}$$

By chasing $g_{\#}$ around the commutative square

$$\begin{array}{ccccc} & \mathrm{Hom}_Y(g^{*}\mathcal{C}, \mathcal{B}) & \cong & \mathrm{Hom}_Z(\mathcal{C}, g_{*}\mathcal{B}) & \\ \mathrm{Hom}_Y(g^{*}\mathcal{C}, f^{\#}) & \uparrow & & \uparrow & \mathrm{Hom}_Z(\mathcal{C}, g_{*}f^{\#}) \\ & \mathrm{Hom}_Y(g^{*}\mathcal{C}, f_{*}\mathcal{A}) & \cong & \mathrm{Hom}_Z(\mathcal{C}, g_{*}f_{*}\mathcal{A}) & \end{array}$$

we have

$$\begin{array}{ccc} g_{\#} & \longmapsto & g^{\#} \\ \updownarrow & & \updownarrow \\ f^{\#}\circ g_{\#} & \longmapsto & (g_{*}f^{\#})\circ g^{\#} \end{array}$$

Thus $f_{\#} \circ (f^{*}g_{\#})$ corresponds to $(g_{*}f^{\#}) \circ g^{\#}$ under the canonical adjunction $\mathrm{Hom}_X(f^{*}g^{*}\mathcal{C}, \mathcal{A}) \cong \mathrm{Hom}_Z(\mathcal{C}, g_{*}f_{*}\mathcal{A})$. ∎

Given a topological space X, we denote by $\mathfrak{Sh}_{\mathfrak{ab}}(X)$ the subcategory of $\mathfrak{Sh}_{\mathfrak{ab}}$ consisting of all objects of $\mathfrak{Sh}_{\mathfrak{ab}}$ of the form $(X, \mathcal{A})$ and all morphisms of $\mathfrak{Sh}_{\mathfrak{ab}}$ of the form $(f, f^{\#})$ with f being the identity mapping of X onto itself. Note that the category $\mathfrak{Sh}_{\mathfrak{ab}}(X)$ is naturally isomorphic to the dual category of $\mathbf{Sh}_{\mathbf{ab}}(X)$. Thus, given a continuous function $f: X \to Y$ of topological spaces, the direct image functor $f_{*}: \mathbf{Sh}_{\mathbf{ab}}(X) \to \mathbf{Sh}_{\mathbf{ab}}(Y)$ and the inverse image functor $f^{*}: \mathbf{Sh}_{\mathbf{ab}}(Y) \to \mathbf{Sh}_{\mathbf{ab}}(X)$ can be regarded also as functors $f_{*}: \mathfrak{Sh}_{\mathfrak{ab}}(X) \to \mathfrak{Sh}_{\mathfrak{ab}}(Y)$ and $f^{*}: \mathfrak{Sh}_{\mathfrak{ab}}(Y) \to \mathfrak{Sh}_{\mathfrak{ab}}(X)$, respectively.

The disjoint union construction of sheaves of Abelian groups, which is a special case of the so-called gluing construction of sheaves (Hartshorne, 1977, Exercises 1.22 and 2.12 of Chapter II), gives rise to the coproduct construction in the category $\mathfrak{Sh}_{\mathfrak{ab}}$, so that $\mathfrak{Sh}_{\mathfrak{ab}}$ is a category with coproducts. It is easy to see the following result.

Proposition 3.2. The pair $(\mathfrak{Sh}_{\mathfrak{ab}}, \mathfrak{cp}_{\mathfrak{Sh}_{\mathfrak{ab}}})$ is an orthogonal category, where $\mathfrak{cp}_{\mathfrak{Sh}_{\mathfrak{ab}}}$ is the class of coproduct diagrams in the category $\mathfrak{Sh}_{\mathfrak{ab}}$.

In the remainder of this section an arbitrarily chosen manual $\mathfrak{M}$ of Boolean locales shall be fixed. Since the forgetful functor $\mathrm{G}\mathfrak{Sh}_{\mathfrak{ab}} \to \mathfrak{Top}$ is an O-functor, it induces a forgetful functor $\mathfrak{sp}$: $\mathrm{O{=}Sh}(\mathfrak{Sh}_{\mathfrak{ab}}) \to \mathrm{O{=}Sh}(\mathfrak{Top})$ by Proposition 1.6. An object $\mathfrak{A}$ in $\mathrm{O{=}Sh}(\mathfrak{Sh}_{\mathfrak{ab}})$ with $\mathfrak{sp}(\mathfrak{A}) = \mathfrak{X}$ is called an O-sheaf of sheaves of Abelian groups *on* $\mathfrak{X}$. Given an object $\mathfrak{X}$ in $\mathrm{O{=}Sh}(\mathfrak{Top})$, the subcategory of $\mathrm{O{=}Sh}(\mathfrak{Sh}_{\mathfrak{ab}})$ consisting of all the objects $\mathfrak{A}$ in $\mathrm{O{=}Sh}(\mathfrak{Sh}_{\mathfrak{ab}})$ with $\mathfrak{sp}(\mathfrak{A}) = \mathfrak{X}$ and all the morphisms $\mathfrak{f}$ in $\mathrm{O{=}Sh}(\mathfrak{Sh}_{\mathfrak{ab}})$ with $\mathfrak{sp}(\mathfrak{f}) = 1_{\mathfrak{X}}$ is denoted by $\mathrm{O{=}Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{X})$. Its dual category is denoted by $\mathrm{Sh}_{\mathrm{ab}}(\mathfrak{X})$.

Lemma 3.3. Given a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha_1} & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_2 & \xrightarrow[\alpha_2]{} & Y_2 \end{array}$$

in the category $\mathfrak{Top}$ and a morphism $(f, f^{\#})$: $(X_1, \mathscr{A}_1) \to (X_2, \mathscr{A}_2)$ of the category $\mathfrak{Sh}_{\mathfrak{ab}}$, there exists a unique morphism $(g, g^{\#})$: $(Y_1, \alpha_{1*}\mathscr{A}_1) \to (Y_2, \alpha_{2*}\mathscr{A}_2)$ of $\mathfrak{Sh}_{\mathfrak{ab}}$ making the diagram of $\mathfrak{Sh}_{\mathfrak{ab}}$

$$\begin{array}{ccc} (X_1, \mathscr{A}_1) & \xrightarrow{(\alpha_1, \alpha_1^{\#})} & (Y_1, \alpha_{1*}\mathscr{A}_1) \\ {\scriptstyle (f,f^{\#})}\downarrow & & \downarrow{\scriptstyle (g,g^{\#})} \\ (X_2, \mathscr{A}_2) & \xrightarrow[(\alpha_2, \alpha_2^{\#})]{} & (Y_2, \alpha_{2*}\mathscr{A}_2) \end{array}$$

commutative, where $\alpha_i^{\#}$: $\alpha_{i*}\mathscr{A}_i \to \alpha_{i*}\mathscr{A}_i$ is an identity homomorphism of sheaves of Abelian groups on Y_i $(i = 1, 2)$. Explicitly it should be the case that $g^{\#} = \alpha_{2*}f^{\#}$.

Proof. It is easy to see that

$$(\alpha_2, \alpha_2^{\#}) \circ (f, f^{\#}) = (\alpha_2 \circ f, \alpha_{2*}f^{\#}) \tag{3.1}$$

It is also easy to see that for any morphism $(g, g^{\#})$: $(Y_1, \alpha_{1*}\mathscr{A}_1) \to (Y_2, \alpha_{2*}\mathscr{A}_2)$ of $\mathfrak{Sh}_{\mathfrak{ab}}$,

$$(g, g^{\#}) \circ (\alpha_1, \alpha_1^{\#}) = (g \circ \alpha_1, g^{\#}) \tag{3.2}$$

Thus $(g, g^{\#})$ makes the latter diagram commutative iff $g^{\#} = \alpha_{2*}f^{\#}$. ∎

We write $\Phi^{\#}(f^{\#}; f, g, \alpha_1, \alpha_2)$ for $g^{\#}$ in the above lemma. The following lemma should be obvious.

Lemma 3.4. Let

$$\{(X_\lambda, \mathscr{A}_\lambda) \xrightarrow{(f_\lambda, f_\lambda^\#)} (X, \mathscr{A})\}_{\lambda \in \Lambda}$$

be an orthogonal sum diagram in $\mathfrak{Sh}_{\mathfrak{ab}}$ and $\{Y_\lambda \xrightarrow{g_\lambda} Y\}_{\lambda \in \Lambda}$ an orthogonal diagram in $\mathfrak{Top}$. Let $\alpha: X \to Y$ and $\alpha_\lambda: X_\lambda \to Y_\lambda$ ($\lambda \in \Lambda$) be morphisms in $\mathfrak{Top}$ with $\alpha \circ f_\lambda = g_\lambda \circ \alpha_\lambda$ for any $\lambda \in \Lambda$. Then

$$\{(Y_\lambda, \alpha_{\lambda *}\mathscr{A}_\lambda) \xrightarrow{(g_\lambda, \phi^\#(f_\lambda^\#; f_\lambda, g_\lambda, \alpha_\lambda, \alpha))} (Y, \alpha_* \mathscr{A})\}_{\lambda \in \Lambda}$$

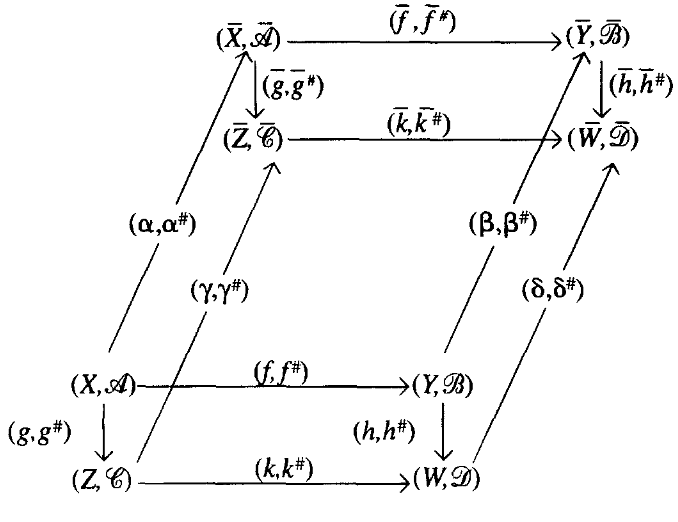
is an orthogonal diagram in $\mathfrak{Sh}_{\mathfrak{ab}}$.

By Lemmas 3.3 and 3.4 it is easy to see the following result.

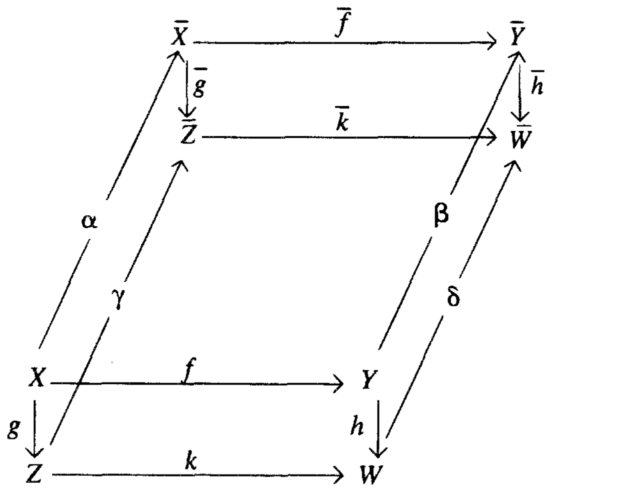
Proposition 3.5. Given a morphism $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ of O-sheaves of topological spaces and an O-sheaf $\mathfrak{A}$ of sheaves of Abelian groups on $\mathfrak{X}$, there exists a unique O-sheaf $\mathfrak{B}$ of sheaves of Abelian groups on $\mathfrak{Y}$ such that $\mathfrak{B}(\mathbf{X}) = \mathfrak{f}(\mathbf{X})_* \mathfrak{A}(\mathbf{X})$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, and the assignment to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of the pair $(\mathfrak{f}(\mathbf{X}), \mathfrak{f}^\#(\mathbf{X}))$ with $\mathfrak{f}^\#(\mathbf{X}): \mathfrak{f}(\mathbf{X})_* \mathfrak{A}(\mathbf{X}) \to \mathfrak{f}(\mathbf{X})_* \mathfrak{A}(\mathbf{X})$ being an identity homomorphism of sheaves of Abelian groups on the topological space $\mathfrak{X}(\mathbf{X})$ is a morphism of O-sheaves of sheaves of Abelian groups from $\mathfrak{A}$ to $\mathfrak{B}$.

The O-sheaf $\mathfrak{B}$ of sheaves of Abelian groups, whose unique existence is guaranteed by the above proposition, is denoted by $\mathfrak{f}_* \mathfrak{A}$ and is called the *direct image of* $\mathfrak{A}$ *under* $\mathfrak{f}$. It remains to make $\mathfrak{f}_*$ a functor from the category $\mathbb{O}$=$\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}})$ to the category $\mathbb{O}$=$\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}})$.

Lemma 3.6. Consider the following cubic diagram of sheaves of Abelian groups, where $\bar{\mathscr{A}} = \alpha_* \mathscr{A}$, $\bar{\mathscr{B}} = \beta_* \mathscr{B}$, $\bar{\mathscr{C}} = \gamma_* \mathscr{C}$, $\bar{\mathscr{D}} = \delta_* \mathscr{D}$, and all homomorphisms of sheaves of Abelian groups $\alpha^\#: \alpha_* \mathscr{A} \to \alpha_* \mathscr{A}$, $\beta^\#: \beta_* \mathscr{B} \to \beta_* \mathscr{B}$, $\gamma^\#: \gamma_* \mathscr{C} \to \gamma_* \mathscr{C}$, and $\delta^\#: \delta_* \mathscr{D} \to \delta_* \mathscr{D}$ are identity homomorphisms:



Suppose that all the faces of the above cubic diagram except the back face are commutative. Suppose also that the following underlying cubic diagram of topological spaces is commutative:



Then the back face of the upper diagram is also commutative.

Proof. We have

$$\begin{aligned}(\bar{h}, \bar{h}^\#) \circ (\bar{f}, \bar{f}^\#) &= (\bar{h} \circ \bar{f}, (\bar{h}_* \beta_* f^\#) \circ (\delta_* h^\#)) \qquad \text{(Lemma 3.3)}\\ &= (\bar{h} \circ \bar{f}, (\delta_* h_* f^\#) \circ (\delta_* h^\#)) \qquad (\bar{h} \circ \beta = \delta \circ h)\\ &= (\bar{h} \circ \bar{f}, \delta_*((h_* f^\#) \circ h^\#))\\ &= (\bar{k} \circ \bar{g}, \delta_*((h_* f^\#) \circ h^\#)) \qquad (\bar{h} \circ \bar{f} = \bar{k} \circ \bar{g})\\ &= (\bar{k} \circ \bar{g}, \delta_*((k_* g^\#) \circ k^\#))\\ &\qquad [(h, h^\#) \circ (f, f^\#) = (k, k^\#) \circ (g, g^\#)]\\ &= (\bar{k} \circ \bar{g}, (\delta_* k_* g^\#) \circ (\delta_* k^\#))\\ &= (\bar{k} \circ \bar{g}, (\bar{k}_* \gamma_* g^\#) \circ (\delta_* k^\#)) \qquad (\bar{k} \circ \gamma = \delta \circ k)\\ &= (\bar{k}, \bar{k}^\#) \circ (\bar{g}, \bar{g}^\#) \qquad \text{(Lemma 3.3)} \quad \blacksquare\end{aligned}$$

The above lemma gives the following forthwith.

Proposition 3.7. Given a morphism $\mathfrak{f}: \mathfrak{X} \to \mathfrak{D}$ in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Top})$ and a morphism $\mathfrak{u}: \mathfrak{A} \to \mathfrak{B}$ in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{X})$, there exists a unique morphism $\mathfrak{v}: \mathbf{f}_*\mathfrak{A} \to \mathfrak{f}_*\mathfrak{B}$ in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{D})$ with $\mathfrak{v}(\mathbf{X}) = \mathfrak{f}_* u(\mathbf{X})$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$.

Thus the morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{D}$ induces a functor $\mathfrak{f}_*$ from the category $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{X})$ to the category $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{D})$. The functor $\mathfrak{f}_*$ can be regarded as a functor from the category $\mathbb{Sh}_{ab}(\mathfrak{X})$ to the category $\mathbb{Sh}_{ab}(\mathfrak{D})$.

The discussion from Lemma 3.3 through Proposition 3.7 can be dualized. By way of example, Lemma 3.3 can be dualized as follows.

Lemma 3.8. Given a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\alpha_1} & Y_1 \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
X_2 & \xrightarrow[\alpha_2]{} & Y_2
\end{array}
$$

in the category $\mathfrak{Top}$ and a morphism $(g, g_\#)\colon (Y_1, \mathcal{B}_1) \to (Y_2, \mathcal{B}_2)$ of the category $\mathfrak{Sh}_{\mathfrak{ab}}$, there exists a unique morphism $(f, f_\#)\colon (X_1, \alpha_1^*\mathcal{B}_1) \to (X_2, \alpha_2^*\mathcal{B}_2)$ of $\mathfrak{Sh}_{\mathfrak{ab}}$ making the diagram of $\mathfrak{Sh}_{\mathfrak{ab}}$

$$
\begin{array}{ccc}
(X_1,\alpha_1^*\mathcal{B}_1) & \xrightarrow{(\alpha_1,\alpha_{1\#})} & (Y_1,\mathcal{B}_1) \\
{\scriptstyle (f,f_\#)}\downarrow & & \downarrow{\scriptstyle (g,g_\#)} \\
(X_2,\alpha_2^*\mathcal{B}_2) & \xrightarrow[(\alpha_2,\alpha_{2\#})]{} & (Y_2,\mathcal{B}_2)
\end{array}
$$

commutative. Explicitly it should be the case that $f_\# = \alpha_1^* g_\#$.

Proof. Since $\alpha_{1\#}$ is the identity homomorphism $\alpha_1^*\mathcal{B}_1 \to \alpha_1^*\mathcal{B}_1$ of sheaves of Abelian groups on X_1 by assumption, we have that

$$(g, g_\#) \circ (\alpha_1, \alpha_{1\#}) = (g \circ \alpha_1, \alpha_1^* g_\#) \tag{3.3}$$

Similarly, since $\alpha_{2\#}$ is the identity homomorphism $\alpha_2^*\mathcal{B}_2 \to \alpha_2^*\mathcal{B}_2$ of sheaves of Abelian groups on X_2 by assumption, we have for any morphism $(f, f_\#)$: $(X_1, \alpha_1^*\mathcal{B}_1) \to (X_2, \alpha_2^*\mathcal{B}_2)$ of $\mathfrak{Sh}_{\mathfrak{ab}}$ that

$$(\alpha_2, \alpha_{2\#}) \circ (f, f_\#) = (\alpha_2 \circ f, f_\#) \tag{3.4}$$

Thus $(f, f_\#)$ makes the latter diagram commutative iff $f_\# = \alpha_1^* g_\#$. ■

We write $\Psi_\#(g_\#; f, g, \alpha_1, \alpha_2)$ for $f_\#$ in the above lemma. This dualization of Lemma 3.3 together with the dualization of Lemma 3.4 naturally leads to the following dualization of Proposition 3.5.

Proposition 3.9. Given a morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{D}$ in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Top})$ and an $\mathbb{O}$-sheaf $\mathfrak{B}$ of sheaves of Abelian groups on $\mathfrak{D}$, there exists a unique $\mathbb{O}$-sheaf

$\mathfrak{B}$ of sheaves of Abelian groups on $\mathfrak{X}$ such that $\mathfrak{A}(\mathbf{X}) = \mathfrak{f}(\mathbf{X})^*\mathfrak{B}(\mathbf{X})$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, and the assignment to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of the pair $(\mathfrak{f}(\mathbf{X}), \mathfrak{f}_{\#}(\mathbf{X}))$ with $\mathfrak{f}_{\#}(\mathbf{X})$: $\mathfrak{f}(\mathbf{X})^*\mathfrak{B}(X) \to \mathfrak{f}(\mathbf{X})^*\mathfrak{B}(\mathbf{X})$ being an identity homomorphism of sheaves of Abelian groups on the topological space $\mathfrak{X}(\mathbf{X})$ is a morphism of O-sheaves of sheaves of Abelian groups from $\mathfrak{A}$ to $\mathfrak{B}$.

Similarly, Lemma 3.6 can readily be dualized, which leads at once to the following dualization of Proposition 3.7.

Proposition 3.10. Given a morphism $\mathfrak{f}$: $\mathfrak{X} \to \mathfrak{Y}$ in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Top})$ and a morphism $\mathfrak{v}$: $\mathfrak{A} \to \mathfrak{B}$ in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{Y})$, there exists a unique morphism $\mathfrak{u}$: $\mathfrak{f}^*\mathfrak{A} \to \mathfrak{f}^*\mathfrak{B}$ in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{X})$ with $u(\mathbf{X}) = \mathfrak{f}^*\mathfrak{v}(\mathbf{X})$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$.

Thus the morphism $\mathfrak{f}$: $\mathfrak{X} \to \mathfrak{Y}$ induces a functor $\mathfrak{f}^*$ from the category $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{Y})$ to the category $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sh}_{\mathfrak{ab}}; \mathfrak{X})$. The functor $\mathfrak{f}_*$ can be regarded as a functor from the category $\mathbb{Sh}_{ab}(\mathfrak{Y})$ to the category $\mathbb{Sh}_{ab}(\mathfrak{X})$.

Lemma 3.11. Given a commutative square of continuous functions

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha_1} & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_2 & \xrightarrow[\alpha_2]{} & Y_2 \end{array}$$

and a morphism $(f, f^{\#})$: $(X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$ in $\mathfrak{Sh}_{\mathfrak{ab}}$, the square

$$\begin{array}{ccc} (X_1,\alpha_1^*\alpha_{1*}\mathcal{A}_1) & \xleftarrow{(1_{X_1},\epsilon_{\alpha_1}(\mathcal{A}_1))} & (X_1,\mathcal{A}_1) \\ {\scriptstyle (f,h^{\#})}\downarrow & & \downarrow{\scriptstyle (f,f^{\#})} \\ (X_2,\alpha_2^*\alpha_{2*}\mathcal{A}_2) & \xleftarrow[(1_{X_2},\epsilon_{\alpha_2}(\mathcal{A}_2))]{} & (X_2,\mathcal{A}_2) \end{array}$$

in $\mathfrak{Sh}_{\mathfrak{ab}}$ is commutative, where $g^{\#} = \Phi^{\#}(f^{\#}; f, g, \alpha_1, \alpha_2)$ and $h_{\#} = \Psi_{\#}(g_{\#}; f, g, \alpha_1, \alpha_2)$.

Proof. By chasing $1_{\alpha_{2*}\mathcal{A}_2}$ around the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{Y_2}(\alpha_{2*}\mathcal{A}_2,\alpha_{2*}\mathcal{A}_2) & \cong & \mathrm{Hom}_{X_2}(\alpha_2^*\alpha_{2*}\mathcal{A}_2,\mathcal{A}_2) \\ {\scriptstyle \mathrm{Hom}_{Y_2}(\alpha_{2*}\mathcal{A}_2,\alpha_{2*}f^{\#})}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{X_2}(\alpha_2^*\alpha_{2*}\mathcal{A}_2,f^{\#})} \\ \mathrm{Hom}_{Y_2}(\alpha_{2*}\mathcal{A}_2,\alpha_{2*}f_*\mathcal{A}_1) & \cong & \mathrm{Hom}_{X_2}(\alpha_2^*\alpha_{2*}\mathcal{A}_2,f_*\mathcal{A}_1) \end{array}$$

we have

$$\begin{array}{ccc} 1_{\alpha_{2*}\mathscr{A}_2} & \longmapsto & \epsilon_{\alpha_2}(\mathscr{A}_2) \\ \big\downarrow & & \big\downarrow \\ \alpha_{2*}f^{\#} & \longmapsto & f^{\#}\circ\epsilon_{\alpha_2}(\mathscr{A}_2) \end{array}$$

Since $\alpha_{2*}f^{\#} = g^{\#}$ by Lemma 3.3, $f^{\#} \circ \epsilon_{\alpha_2}(\mathscr{A}_2)$ corresponds to $g^{\#}$ under the canonical adjunction

$$\mathrm{Hom}_{X_2}(\alpha_2^*\alpha_{2*}\mathscr{A}_2, f_*\mathscr{A}_1) \cong \mathrm{Hom}_{Y_2}(\alpha_{2*}\mathscr{A}_2,\ \alpha_{2*}f_*\mathscr{A}_1)$$

By chasing $1_{\alpha_{1*}\mathscr{A}_1}$ around the commutative square

$$\begin{array}{ccccc} & \mathrm{Hom}_{Y_1}(g^*\alpha_{2*}\mathscr{A}_2,\alpha_{1*}\mathscr{A}_1) & \cong & \mathrm{Hom}_{X_1}(\alpha_1^*g^*\alpha_{2*}\mathscr{A}_2,\mathscr{A}_1) & \\ \mathrm{Hom}_{Y_1}(g_{\#},\alpha_{1*}\mathscr{A}_1) & \big\uparrow & & \big\uparrow & \mathrm{Hom}_{X_1}(\alpha_1^*g_{\#},\mathscr{A}_1) \\ & \mathrm{Hom}_{Y_1}(\alpha_{1*}\mathscr{A}_1,\alpha_{1*}\mathscr{A}_1) & \cong & \mathrm{Hom}_{X_1}(\alpha_1^*\alpha_{1*}\mathscr{A}_1,\mathscr{A}_1) & \end{array}$$

we have

$$\begin{array}{ccc} g_{\#} & \longmapsto & \epsilon_{\alpha_1}(\mathscr{A}_1)\circ(\alpha_1^*g_{\#}) \\ \big\uparrow & & \big\uparrow \\ 1_{\alpha_{1*}\mathscr{A}_1} & \longmapsto & \epsilon_{\alpha_1}(\mathscr{A}_1) \end{array}$$

Since $\alpha_1^*g_{\#} = h_{\#}$ by Lemma 3.8, $g_{\#}$ corresponds to $\epsilon_{\alpha_1}(\mathscr{A}_1) \circ h_{\#}$ under the canonical adjunction

$$\mathrm{Hom}_{Y_1}(g^*\alpha_{2*}\mathscr{A}_2,\ \alpha_{1*}\mathscr{A}_1) \cong \mathrm{Hom}_{X_1}(\alpha_1^*g^*\alpha_{2*}\mathscr{A}_2,\ \mathscr{A}_1)$$

By Corollary 2.2 the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{X_1}(\alpha_1^*g^*\alpha_{2*}\mathscr{A}_2,\mathscr{A}_1) & \cong & \mathrm{Hom}_{Y_1}(g^*\alpha_{2*}\mathscr{A}_2,\alpha_1\mathscr{A}_1) \\ \| & & \wr\| \\ \mathrm{Hom}_{X_1}(f^*\alpha_2^*\alpha_{2*}\mathscr{A}_2,\mathscr{A}_1) & & \mathrm{Hom}_{Y_2}(\alpha_{2*}\mathscr{A}_2,g_*\alpha_{1*}\mathscr{A}_1) \\ \wr\| & & \| \\ \mathrm{Hom}_{X_2}(\alpha_2^*\alpha_{2*}\mathscr{A}_2,f_*\mathscr{A}_1) & \cong & \mathrm{Hom}_{Y_2}(\alpha_{2*}\mathscr{A}_2,\alpha_{2*}f_*\mathscr{A}_1) \end{array}$$

By chasing $f^{\#} \circ \epsilon_{\alpha_2}(\mathscr{A}_2)$ around this commutative diagram, we have

$$\begin{array}{ccc} \epsilon_{\alpha_1}(\mathcal{A}_1)\circ h_\# & \longleftarrow\!\!\dashv & g_\# \\ \downarrow & & \uparrow \\ f^\#\circ\epsilon_{\alpha_2}(\mathcal{A}_2) & \longmapsto & g^\# \end{array}$$

Thus $f^\# \circ \epsilon_{\alpha_2}(\mathcal{A}_2)$ corresponds to $\epsilon_{\alpha_1}(\mathcal{A}_1) \circ h_\#$ under the canonical adjunction $\mathrm{Hom}_{X_2}(\alpha_2^\ddagger\alpha_{2*}\mathcal{A}_2, f_*\mathcal{A}_1) \cong \mathrm{Hom}_{X_1}(f^*\alpha_2^\ddagger\alpha_{2*}\mathcal{A}_2, \mathcal{A}_1)$. Since the upper representation of $(1_{X_2}, \epsilon_{\alpha_2}(\mathcal{A}_2)) \circ (f, f^\#)$ is $(f, f^\# \circ \epsilon_{\alpha_2}(\mathcal{A}_2))$ and the lower representation of $(f, h_\#) \circ (1_{X_1}, \epsilon_{\alpha_1}(\mathcal{A}_1)) = (f, \epsilon_{\alpha_1}(\mathcal{A}_1) \circ h_\#)$, the proof is complete. ■

By the above lemma it is easy to see the following result.

Proposition 3.12. Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{D}$ be a morphism in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Top})$. Let $\mathfrak{A}$ be an object in $\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{X})$. Then the assignment to each object $\mathbf{X}$ in $\mathfrak{M}$ of $(1_{\mathfrak{X}(\mathbf{X})}, \epsilon_{\mathfrak{f}(\mathbf{X})}(\mathfrak{A}(\mathbf{X})))$ is a morphism $\bar{\epsilon}_{\mathfrak{f}}(\mathfrak{A}): \mathfrak{A} \to \mathfrak{f}^*\mathfrak{f}_*\mathfrak{A}$ in $\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{X})$, so that the assignment to each object $\mathfrak{A}$ in $\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{X})$ of $\bar{\epsilon}_{\mathfrak{f}}(\mathfrak{A})$ is a natural transformation

$$\bar{\epsilon}_{\mathfrak{f}}: I_{\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{X})} \stackrel{\cdot}{\to} \mathfrak{f}^*\mathfrak{f}_*$$

The dualization of $\bar{\epsilon}_{\mathfrak{f}}$ is denoted by $\epsilon_{\mathfrak{f}}$, which is a natural transformation from $\mathfrak{f}^*\mathfrak{f}_*$ to $I_{\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{X})}$.

Lemma 3.11 can be dualized, which leads to the following dualization of Proposition 3.12.

Proposition 3.13. Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{D}$ be a morphism in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Top})$. Let $\mathfrak{B}$ be an object in $\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{D})$. Then the assignment to each object $\mathbf{X}$ in $\mathfrak{M}$ of $(1_{\mathfrak{D}(\mathbf{X})}, \epsilon_{\mathfrak{f}(\mathbf{X})}(\mathfrak{A}(\mathbf{X})))$ is a morphism $\bar{\mathfrak{n}}_{\mathfrak{f}}(\mathfrak{B}): \mathfrak{f}_*\mathfrak{f}^*\mathfrak{B} \to \mathfrak{B}$ in $\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{D})$, so that the assignment to each object $\mathfrak{B}$ in $\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{D})$ of $\bar{\mathfrak{n}}_{\mathfrak{f}}(\mathfrak{B})$ is a natural transformation

$$\bar{\mathfrak{n}}_{\mathfrak{f}}: \mathfrak{f}^*\mathfrak{f}_* \stackrel{\cdot}{\to} I_{\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{D})}$$

The dualization of $\bar{\mathfrak{n}}_{\mathfrak{f}}$ is denoted by $\mathfrak{n}_{\mathfrak{f}}$, which is a natural transformation of $I_{\mathfrak{Sh}_{\mathfrak{ab}}(\mathfrak{D})}$ to $\mathfrak{f}^*\mathfrak{f}_*$.

Now we are ready to present the main result of this section.

Theorem 3.14. Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{D}$ be a morphism in $\mathbb{O}=\mathbb{Sh}(\mathfrak{Top})$. Then $\langle\mathfrak{f}^*, \mathfrak{f}_*, \mathfrak{n}_{\mathfrak{f}}, \epsilon_{\mathfrak{f}}\rangle$ is an adjunction from $\mathbf{Sh}_{\mathbf{ab}}(\mathfrak{D})$ to $\mathbf{Sh}_{\mathbf{ab}}(\mathfrak{X})$.

Proof. It suffices to note that the triangular identities of the canonical adjunctions corresponding to $\mathfrak{f}(\mathbf{X})$ for all objects $\mathbf{X}$ in $\mathfrak{M}$ yield the desired triangular identities (MacLane, 1971, p. 83). ■

4. EMPIRICAL SCHEME THEORY

Throughout this section we assume that $\mathfrak{M}$ is a *finite* manual of Boolean locales. The principal concern of this section is to generalize the adjunction between the category $\mathfrak{Sch}$ of schemes and the category $\mathfrak{RLoc}$ of ring locales (Hartshorne, 1977, Chapter II, Exercise 2.4) to that between $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sch})$ and $\mathbb{O}=\mathbb{Sh}(\mathfrak{RLoc})$, where the category $\mathfrak{Sch}$ can be regarded as an orthogonal category by taking the coproduct diagrams in $\mathfrak{Sch}$ as orthogonal sum diagrams as in $\mathfrak{Sh}_{\mathfrak{ab}}$.

The following proposition is well known, for which the reader is referred, e.g., to Hungerford (1974, Chapter 3, Section 2, Exercise 22).

Proposition 4.1. If R is a ring of the form $R_1 \times \cdots \times R_n$ for some rings $R_1, \ldots, R_n$, then any ideal I of R is of the form $I_1 \times \cdots \times I_n$, where I_i is an ideal of R_i $(1 \le i \le n)$.

Corollary 4.2. If R is a ring of the form $R_1 \times \cdots \times R_n$ for some rings $R_1, \ldots, R_n$, then any prime ideal I of R is of the form $I_1 \times \cdots \times I_n$, where $I_i = R_i$ for all i $(1 \le i \le n)$ except exactly one i_0, and I_{i_0} is a prime ideal of R_{i_0}.

Assigning to each ring of its spectrum gives a functor $\mathfrak{Spec}$: $\mathfrak{RLoc} \to \mathfrak{Sch}$, for which we have the following result.

Proposition 4.3. The functor $\mathfrak{Spec}$: $\mathfrak{RLoc} \to \mathfrak{Sch}$ preserves finite orthogonal diagrams.

Proof. If a ring R is of the form $R_1 \times \cdots \times R_n$ for some rings $R_1, \ldots, R_n$, then the underlying topological space of $\mathfrak{Spec}(R)$ is easily seen, by the above proposition and its corollary, to be the topological sum of the underlying topological spaces of $\mathfrak{Spec}(R_1), \ldots, \mathfrak{Spec}(R_n)$. Furthermore, it is also easy to see that if I is a prime ideal of R of the form $I_1 \times \cdots \times I_n$ with $I_i = R_i$ for all i $(1 \le i \le n)$ except exactly one i_0 and I_{i_0} being a prime ideal of R_{i_0}, then the localization R_I of R with respect to the prime ideal I is naturally isomorphic to the localization $(R_{i_0})_{I_{i_0}}$ of R_{i_0} with respect to the prime ideal I_{i_0}. Thus the desired result should be evident. ■

Since we have assumed that the manual $\mathfrak{M}$ of Boolean locales is finite, the functor $\mathfrak{Spec}$: $\mathfrak{RLoc} \to \mathfrak{Sch}$ naturally induces a functor $\mathbb{Spec}$: $\mathbb{O}=\mathbb{Sh}$ $(\mathfrak{RLoc}) \to \mathbb{O}=\mathbb{Sh}(\mathfrak{SLh})$ by the above proposition and Corollary 1.7.

Taking global sections of the structure sheaves of schemes gives the globalization functor Γ: $\mathfrak{Sch} \to \mathfrak{RLoc}$, for which the following proposition should be obvious.

Proposition 4.4. The functor Γ: $\mathfrak{Sch} \to \mathfrak{RLoc}$ is an O-functor.

By the above proposition and Proposition 1.6 the functor Γ: $\mathfrak{Sch} \rightarrow \mathfrak{RLoc}$ naturally induces a functor Γ_{O}: $\mathbb{O}=\mathbb{Sh}(\mathfrak{Sch}) \rightarrow \mathbb{O}=\mathbb{Sh}(\mathfrak{RLoc})$.

By the same token as in the preceding section, the canonical adjunction

$$\mathrm{Hom}_{\mathfrak{Sch}}(V, \mathfrak{Spec}\, A) \cong \mathrm{Hom}_{\mathfrak{RLoc}}(\Gamma(V), A) \tag{4.1}$$

for a scheme V and a ring A induces the following adjunction.

Theorem 4.5. There is a canonical adjunction

$$\mathrm{Hom}_{\mathrm{O}=\mathbb{Sh}(\mathfrak{Sch})}(\mathfrak{V}, \mathbb{Spec}(\mathfrak{R})) \cong \mathrm{Hom}_{\mathrm{O}=\mathbb{Sh}(\mathfrak{RLoc})}(\Gamma_{\mathrm{O}}(\mathfrak{V}), \mathfrak{R})$$

for an O-sheaf $\mathfrak{V}$ of schemes and an O-sheaf $\mathfrak{R}$ of ring locales.

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